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Research Article

Eighth-grades students' mental models in solving a number pattern problem

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Abstract

This study aims to explore all the types of students' mental models of number patterns. The study used a qualitative approach with an explorative type. The subjects used to characterize the student's mental models in this study were 46 eighth grade students in Indonesia. To reveal the subjects' mental model, they were asked to solve the number pattern problem and were interviewed. For ensuring the validity and reliability of the research results, triangulation technique was used by comparing the results of video recording interviews and written test results. The study showed that in solving the problem of number patterns given, there were 4 types of mental models. They were formal direct mental model, formal indirect mental model, synthetic direct mental model, and synthetic indirect mental model. What we found in this study shows that some students have different mental models to solve the problem. Hence, in future teachers must introduce various strategies to solve the problem and conduct learning that can enrich students' mental models.

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Introduction

Problems solving is one of the five standards of the school's mathematical processes and research (National Council of Teachers of Mathematics [NCTM], 2000). NCTM (2000) also stated that problem solving is an application that should be realized through a mathematical curriculum to provide context for the learning and application of mathematical ideas. Indonesia's 2013 curriculum also requires students to be able to independently, creatively, and proficiently develop knowledge in solving mathematics problems (Suryani et al. 2020). Mathematical problem solving skills are not merely skills taught and used in Mathematics, but also are basic skills used in students' daily lives.

One of the goals of school mathematics learning is the ability to solve problems (Goodnough & Hung, 2008; Ministry of Education and Culture, 2016; Putra et al. 2017). Therefore, problem solving has become an important focus in school mathematics curriculum started from primary school level to the high school level (Rogers et al. 2011). The ability to solve mathematical problems is the ability of students to overcome the problems that were not clear the answer. Problems that arise in solving problems are ways that students use in solving mathematical problems has not been systematic or sequential, so the ability of students in solving math problems has not been maximized (Widodo et al. 2018).

Patterns provide basis for Mathematical thinking. According to Vogel (2005), analyse and describing pattern and its nature become one of the goals of Mathematics. Mulligan (2002) said that almost all Mathematics is based on patterns and structures. Therefore, many mathematicians declare Mathematics as "the science of patterns" (Resnik, 1997; Tikekar, 2009).

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Furthermore, Radford (2006) explained that when someone facing generalization patterns problems, she/he will attempt to solve the problem of pattern generalization using different strategies. The strategies used can be differentiated into naive induction and generalization. Naive induction is the generalization of patterns that are based on trial and error. Furthermore, generalization consists of the generalization of Arithmetic and the generalization of algebraic. The generalization of algebraic consists of factual, contextual, and symbolic generalizations. The factual type of generalization is a generalized type based on known facts. A contextual generalization is a generalized type based on the context of the problem and is limited to particular objects. The symbolic generalization is a generalization type associated with an algebraic object or symbol that is not limited to particular objects.

Generalization, according to Radford (2006), can be viewed as a mathematical representation of a person at the time of solving the problem of generalization of patterns (Prayekti et al. 2019). NCTM (2000) said that "representing involves translating a problem or an idea into a new form. Representing includes the translation of diagram or physical model into symbols or words. It is also used in translating or analysing verbal problems to make the meaning clear." By accurate representation, students can develop their mathematical problem solving skills (Rofiki & Santia, 2018). As such, mathematical representations are depictions, translations, disclosures, reappointment, symbolism or even modelling of ideas, thoughts, mathematical concepts, and inter-relation among which are contained in a configuration, construction, or situation of specific problems that students display in diverse forms in an effort to gain clarity of meaning, demonstrate understanding, find solutions to the problems faced, or build mental model.

According to Chi (2008), mental model is the internal representation of a concept, or a system of inter-related concepts. referencesJansoon, Cool & Somsook (2009) stated that mental model represents the mind of each individual used to describe and explain a concept when learning. Mental Model is an overview of the concepts that a student has in mind to explain a situation or process that is taking place (Greca & Moreira, 2002). Based on the description, mental model is an internal representation that can illustrate the student's knowledge structure and explain how the problem solving process is done. The student's mental model needs to be diagnosed in learning since it gives educators information about the quality and quantity of the student's concepts that they can help students to develop their conceptual skill (Chiou, 2013). This is because the mental model plays role in explaining individual reasoning when attempting to understand, predict, or explain a concept (Fazio et al. 2013).

The construction of student's mental model is based on experiences and needs of making predictions and resolving problems in learning (Halim et al. 2013). Chittleborough (2004) also said that the mental model is essential for making predictions and solving mathematical problems. Therefore, when students have a mental model of a concept that is in accordance with the formal concept, the student, then, will be able to make a good explanation of the problem solving about the concept, otherwise if the student has a wrong or inappropriate mental model, then the student will have difficulties in resolving the problems or even have a missed conception on the concept. Vosniadou and Brewer (1992) identified three categories in describing students' mental models from a concept: initial, synthetic, and formal.

Some researches on mental models have been conducted, including the one did by Bofferding (2014) which reports on the results of his research on elementary school first class students' mental model in understanding sequences, values, and directions of Integer size. In his research, Bofferding (2014) showed that students' mental models about sequence and integer values can be classified into 5 types: initial mental model, transition mental model I, synthetic mental model, transition mental model II, and formal mental model.

A research on the student's mental model in solving patterns generalization problems conducted by Prayekti et al. (2019) suggested that there are two types of mental models when solving the problems of patterns generalization: direct and Indirect mental models. The students are said to have a direct mental model if they only use one type of generalization strategy in problem solving i.e. using algebraic strategy or arithmetic strategy according to Radford (2006), and they are said to have an indirect mental model if the strategy used combining two types of generalization strategy (algebraic and arithmetic strategies) according to Radford (2006).

Based on the previous research have been conducted by several researchers (Bofferding, 2014; Prayekti et al. 2019; Radford, 2006; Vosniadou, 1994; Vosniadou & Brewer, 1992) and the presence of preliminary research findings that students with the same mental model can have different form of generalization. Thus, it is necessary to conduct deeper research to figure out the characteristics of students' mental models based on differences of generalization form used in resolving problems. The researchers wanted to investigate how junior high school students' mental models in solve mathematical problems about number patterns. Student mental models was analysed based on the strategies chosen by students during the problem solving process.

Problem of Study

The study problem was the low skill of students' problem solving in generalizing a number pattern. Also, the studies of students' mental models on a number pattern are very scarce. Yet, mental models play a significant role in students' problem solving skill. Based on the previous studies have been conducted by several scholars (Bofferding, 2014; Prayekti et al., 2019; Radford, 2006; Vosniadou, 1994; Vosniadou & Brewer, 1992) and the presence of preliminary research findings that students with the same mental model can have a different form of generalization. Thus, it is necessary to conduct more in-depth research to detect the characteristics of students' mental models based on various generalization forms. The purpose of this study is to describe the eighth grades students' mental models in solving number pattern problem, based on the strategies chosen by students' during the problem solving process. The main problem of research is;

- How the mental models of eighth graders in solve number pattern problem ?
- What characterize the 8th graders mental models based on the strategies they chosen in solving the number patterns problem ?

Method

Research Design

This research used qualitative research. Creswell (2012) explained about characteristics of qualitative research approach that research process will always dynamically develop. Technique to select the subject was purposive sampling.

Participants

The subjects in this study are eight graders of Junior High School in Indonesia. Participants in the study are 46 eight graders students of Junior High School in Indonesia. There were 19 male and the rest were 27 female. The demography of participants it shown in Table 1.

Table 1.

Demographic Structures of Participants

Characteristics	f	Percentage
Grades 8th	46	100%
Male	19	41,3%
Female	27	58,7%

Data Collection Tool

The students who answered the problem given correctly are grouped into 4 mental model categories according to the strategies they use as shown in Table 2. Here is the distribution of students' response counts shown in Table 3.

Table 2.

The Criteria of Students' Mental Models in Solving Patterns Generalization Problems (Prayekti et al. 2019)

Core Category	Student's Strategy in Solving Pattern Generalization Problems
Direct Mental Model	One strategy (algebraic or arithmetic strategy)
Indirect Mental Model	Combination of algebraic and arithmetic strategies

Table 3.

The Number of Students at Each Mental Model Type in Solving Pattern Generalization Problems with Correct Answers.

The kind of Mental Model in Solving Generalizati0n Problems		CODE	Number of Students
Used Generalization Strategy	Resulted Generalization Form		
Direct Mental Model	Synthetic model	Synthetic Direct Mental Model (SDMM)	10 students
	Formal model	Formal Direct Mental Model (FDMM)	8 students
Indirect Mental Model	Synthetic model	Synthetic Indirect Mental Model (SIMM)	9 students
	Formal model	Formal Indirect Mental Model (FIMM)	5students

Based on the four types of mental model shown in Table 3, one student of each type was selected for interview as subject of this research. The selected subjects were based on the highest students' communication skills of each type of mental models, namely Subject 1 (S1) who represents SDMM, Subject 2 (S2) who represents SIMM, Subject 3 (S3) who represents FDMM, and Subject 4 (S4) who represents FIMM.

In this study, data were collected from the test results. The given problem in this research is presented in Figure 1.

Consider the following sequence of numbers:
1, 3, 6, 11, 20, *s*, 70, *t*, ... ,

Figure 1.
The Problem of Pattern Generalization

Validity of Data Collection Tool

Before the problem given in Figure 1 used in this study, the problem was validated. Researchers used 3 experts for validation. Aspects considered in conducting test validation include (1) the content aspect, which contains 5 question items, (2) the concept aspect, which contain 4 question items, and (3) the language aspect, which contains 3 question items. The complete test validation results are presented in Table 4.

Table 4.
The Results of Expert Validation

Aspect	$\overline{x_1}$	$\overline{x_2}$	$\overline{x_3}$	V(%)
Content	3,43	3,29	3,57	85,71
Concept	3,50	3,25	3,75	87,50
Language	3	3,33	3,33	80,56
		$\overline{V}$(%)		84,59

Based on an average of each aspect, an overall average of 84.59%, and established validity criteria (valid if the overall average is more than 75%), the test prototype meets the validity criteria and does not need to be revised so that it can be directly used for research.

Data Analysis

This research is included in a qualitative research with a descriptive approach. Data were derived from the written answers after students having resolved the problems on pattern generalization concept. The researchers also observed the students' activities when completing the tests provided. Furthermore, the researchers conducted interviews with selected research subjects referring to the written test results to clarify the answers they wrote. In this study, each interview was recorded using the audio-video recorder and the written test results of the student were collected. To ensure the validity and reliability of the research, data analysis was conducted by triangulation through the view of data from the video, transcript of the interview and comparing it with the student's written test data (Golfashani, 2003; Mathison, 1988).

Results

The following are descriptions and explanations of the four types of students' mental model when solving the problem of patterns generalization generated in this study based on written answers and explanations of the students during interviews.

Synthetic Direct Mental Model

The written test result of S1 can be seen in Figure 2.

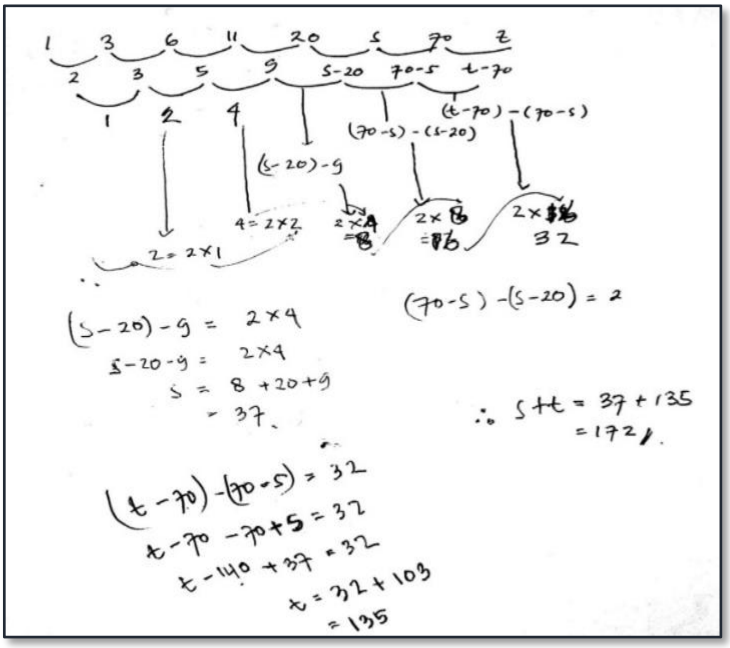


Figure 2.
S1 Written Test Result

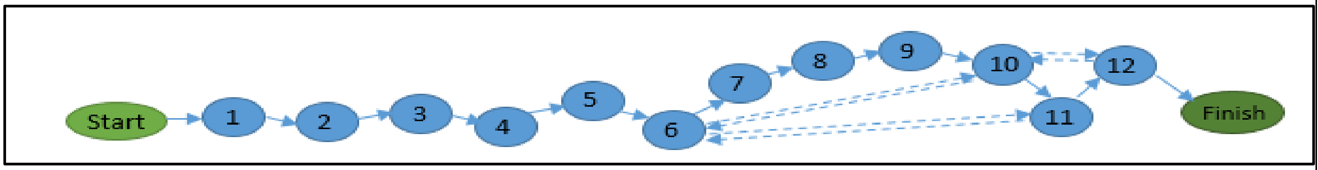
From an interview with S1, it was known that in the beginning S1 opined that to solve a given problem s/he has to look for patterns that are formed from the difference between numbers in a given row. This is shown by the following interview transcript:

Q 04: OK, what do you have in mind when reading this problem for the first time?
S1 04: Eee…..this problem is about the pattern of numbers, Ma'am. The pattern continues to increase according to the difference.
Q 05: What do you mean?
S1 05: Yes…this is from number one to the next number, there must be a difference which always has a pattern. So, the pattern continues to increase, Ma'am.

In order to fix the problem, S1 did the calculation to determine the difference between numbers and then pay attention to the differences obtained, i.e. 2, 3, 5, 9, ..., and considered that the difference is not the same. and came to the conclusion that the differences are not the same that she/he cannot determine the next 3 numbers from rows 2, 3, 5, 9, … each of which showing the difference between s and 20 $(s-20)$, 70 and s $(70-s)$, and t and 70 $(t-70)$. S1 believes that if the pattern of differences obtained is known, then s and t can be known easily.

Furthermore, after experiencing confusion due to not being able to determine the pattern of rows of numbers 2, 3, 5, 9, $(s-20)$, $(70-s)$, $(t-70)$, ..., S1 looked for a pattern of differences between the newly formed new sequence (2, 3, 5, 9, $(s-20)$, $(70-s)$, $(t-70)$, ...). Sequence of differences obtained are 1, 2, 4, $(s-20)-9$, (70-s) - $(70-s)-(s-20)$, $(t-70)-(70-s)$, ... Furthermore, after S1 determined the pattern of numbers 1, 2, 4, $(s-20)-9$, $(70-s)(s-20)$, $(t-70)-(70-s)$, ..., and because the second number of that pattern and so on is twice the previous number, S1 then calculated the values of s and t based on the previous steps. S1 got s and t based on the expected pattern of a sequence 1, 2, 4, $(s-20)-9$, $(70-s)(s-20)$, $(t-70)-(70-s)$, ... S1 noted that $(s-20)-9=2x4$, so $s=37$ and $(t-70)-(70-s)=32$ so $t=135$. After that, S1 proved whether the values of s and t are correct based on the allegation.

S1 thinking process in solving given problems which was based on analysis of the S1 answers and interview results can be described in the S1 thinking process flow in Figure 3.



Notes:

1. Observing the pattern of numbers: 11, 3, 6, 11, 20, s , 70, t , ...
2. Organizing the problem by writing a curved line between two adjacent numbers in rows 1, 3, 6, 11, 20, s , 70, t , ..., which indicates the difference between the two numbers in the row of numbers
3. Writing the difference between two numbers that are close together in rows of numbers 1, 3, 6, 11, 20, s , 70, t , ... under the curve made
4. Paying attention to the difference resulted from the previous step (4), then states that the difference in the sequence of numbers 1, 3, 6, 11, 20, s , 70, t , ..., is not the same, getting bigger, bigger, irregular, and forming the new sequence of numbers, namely: 2, 3, 5, 9, $(s-20)$, $(70-s)$, $(t-70)$, ...
5. Observing the pattern of a number sequence which is the difference between the two numbers in rows 2, 3, 5, 9, $(s-20)$, $(70-s)$, $(t-70)$, ..., namely: 1, 2, 4, $(s-20)-9$, $(70-s)(s-20)$, $(t-70)-(70-s)$, ...
6. Manipulating numbers in rows 1, 2, 4, ... into multiplication or addition of two integers
7. Writing a line / sign that indicates the regularity of number patterns in rows of numbers 1, 2, 4, ... which has been manipulated
8. Stating a row of numbers 1, 2, 4, ... , regularly follow the rule that the next number is 2 times the previous number.
9. Writing down the next number of rows of numbers 1, 2, 4, is 8, 16, 32, ...
10. Calculating s with respect to step 6, so that $(s-20)-9=2x4$ and stating $s=37$
11. Calculating t with respect to step 6, so that: $(t-70)-(70-s)=16x2$, and stating $t=135$
12. Showing that the alleged rule found is correct by showing that $(70-s)(s-20)$is equal to 16 if $s=37$ and write the answers of the problem are $s+t=37+135=172$.

Steps of completion — Coordination

Arithmetic Strategy — Algebraic Strategy

Figure 3.

S1 Thinking Process Flow in Solving Problems

In solving problems, S1 used arithmetic strategy from beginning to the end of completion. The direction of strategy that S1 choosing to answer the problem can be seen from Figure 4.

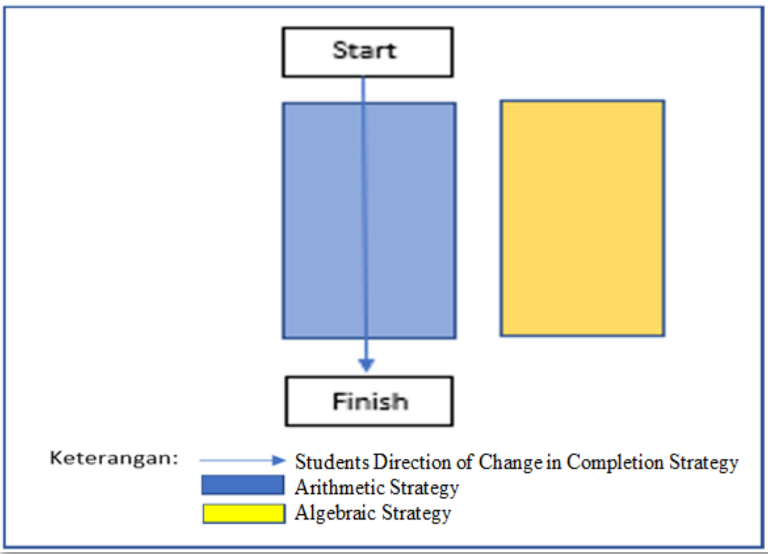
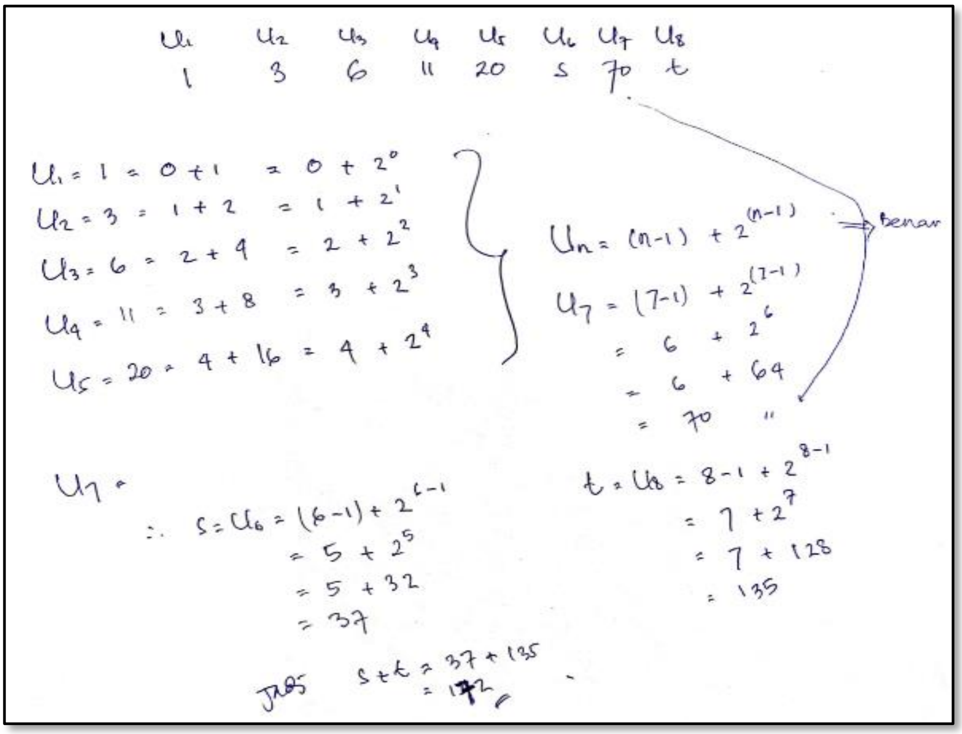


Figure 4.

The Scheme of Changes in the Selection of Generalization Strategies by S1 in Solving Research Problems

Formal Direct Mental Model.

The written test result of S2 can be seen in Figure 5.



U_1 U_2 U_3 U_4 U_5 U_6 U_7 U_8

1 3 6 11 20 s 70 t

$U_1 = 1 = 0 + 1 = 0 + 2^0$

$U_2 = 3 = 1 + 2 = 1 + 2^1$

$U_3 = 6 = 2 + 4 = 2 + 2^2$

$U_4 = 11 = 3 + 8 = 3 + 2^3$

$U_5 = 20 = 4 + 16 = 4 + 2^4$

$U_n = (n-1) + 2^{(n-1)}$ benar

$U_7 = (7-1) + 2^{(7-1)}$

$= 6 + 2^6$

$= 6 + 64$

$= 70$

$U_7 =$

$\therefore s = U_6 = (6-1) + 2^{6-1}$

$= 5 + 2^5$

$= 5 + 32$

$= 37$

$t = U_8 = 8 - 1 + 2^{8-1}$

$= 7 + 2^7$

$= 7 + 128$

$= 135$

Jaws $s + t = 37 + 135$

$= 172$

Figure 5.

S2 Written Test Result

Based on the results of interview with S2, it was known in the initial stages of problem solving S2 opined that in order to solve a given problem, a general number pattern must be sought first and then it can be used to find the desired values. This is shown by the following interview transcript.

Q 04: Then what was your first thought when you first read about this?

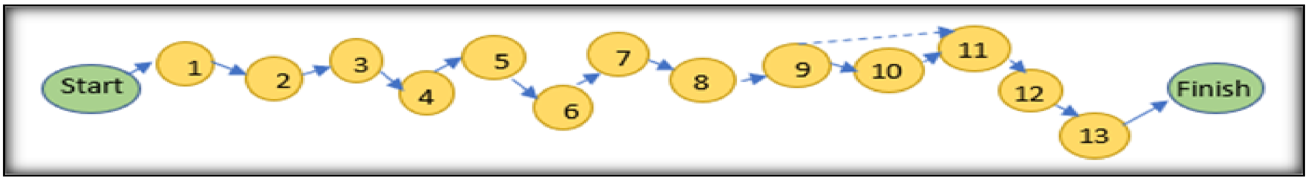
S2 04: This problem is about rows of numbers, Ma'am . Then I was told to look for results from s + t. So because of the sequence of numbers ..., it means you have to find this pattern, Ma'am. It means the general formula, Ma'am ...

Q 05: What do you do then?

S2 05: I first rewrite these numbers but I give the symbols U1, U2. U3 and so on Ma'am ... until U8 is equal to this t Ma'am.

The next step taken by S2 is to make the relationship between variables U_1 and 1, U_2 and 3, U_3 and 6, etc. U_6 and s, U_8 and t. To determine the relationship between numbers, S2 manipulated each numbers in sequence into the sum of two integers. S2 stating $U_1 = 1 = 0 + 1$; $U_2 = 3 = 1 + 2$; $U_3 = 6 = 2 + 4$; $U_4 = 11 = 3 + 8$; $U_5 = 20 = 4 + 16$. Next, after manipulating the numbers in row numbers 1, 3, 6, 11, 20, ..., into the sum of two integers, S2 separated the first set of numbers and the second set of numbers and connected them with the index n. Then S2 predicted the general formula of rows of numbers 1, 3, 6, 11, 20, ... is $U_n = (n-1) + 2^{(n-1)}$. In the next step, S2 showed through calculations that if $n = 7$ and by using the formula $U_n = (n-1) + 2^{(n-1)}$ the results are the same as what is known in the problem that is 70. After verifying and believing that the formula general sequence of numbers $U_n = (n-1) + 2^{(n-1)}$ is true and can be applies to all n natural numbers, S2 then looked for the values s and t and obtained $s = 37$ and $t = 135$ and answered confidently that $s + t = 172$.

Based on the written answers and interview results, it can be seen that the generalization used by S2 to solve the given problem is $U_n = (n-1) + 2^{(n-1)}$. Thus S2 has stated the generalization of numbers given in symbolic formal form. S2 thinking process in solving given problems which was based on analysis of answers and interview results can be illustrated in the S2 thinking process flow in Figure 6.



Notes:

1. Observing the pattern of numbers 1, 3, 6, 11, 20, *s*, 70, *t*,…
2. Writing variables that state the term syllable using symbols, for example U_1, U_2, U_3, etc.
3. Relating variables U_1 and 1, U_2 and 3 , U_3 and 6 , dst U_6 and s, U_8 and t
4. Manipulating numbers in rows 1, 3, 6, 11, 20, ... into a sum of 2 integers
5. Observing the pattern of numbers $U_1 = 1 = 0 + 1$; $U_2 = 3 = 1 + 2$; $U_3 = 6 = 2 + 4$; $U_4 = 11 = 3 + 8$; $U_5 = 20 = 4 + 16$; ....
6. Writing lines/circles/signs that indicate regularity in the pattern of numbers manipulated
7. Making a relationship between n indices stating the order with the pattern found
8. Predicting the number sequence rule from the regularity found $(n-1) + 2^{(n-1)}$
9. Stating the expected general formula for the n-th number of the number line 1, 3, 6, 11, 20, s, 70, t, ... is $U_n = (n-1) + 2^{(n-1)}$ for all n natural numbers
10. Showing through calculations that $U_n = (n-1) + 2^{(n-1)}$ is true for some n,
11. Expressing her/his belief that $U_n = (n-1) + 2^{(n-1)}$is true for all n natural numbers.
12. Calculating s and t using the general formula $U_n = (n-1) + 2^{(n-1)}$that has been found, and declaring $U_6 = s = 37$, and $U_8 = t = 135$
13. Giving an explanation to convince others that $s = 37$and $t = 135$ to $s + t = 172$

Sequenceof Steps — Coordination

Arithmatics Strategy — Algebraic Strategy

Figure 6.

S2 Thinking Process Flow in Solving Problems

In solving problems, S2 used Algebraic strategy from beginning to the end of completion. The direction of strategy that S2 choosing to answer the problem can be seen from Figure 7.

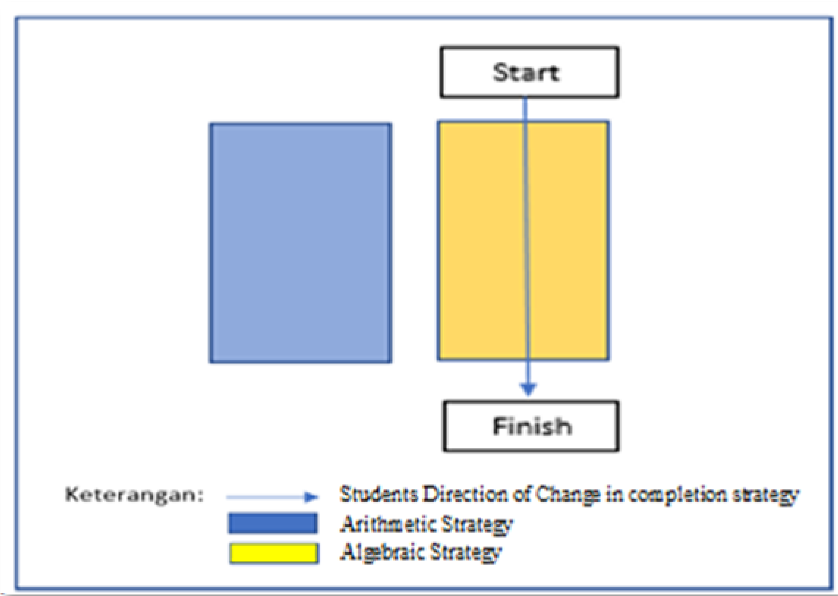


Figure 7.

The Scheme of Changes in the Selection of Generalization Strategies by S2 in Solving Research Problems

Synthetics Indirect Mental Model.

The written test result of S3 can be seen in Figure 8.

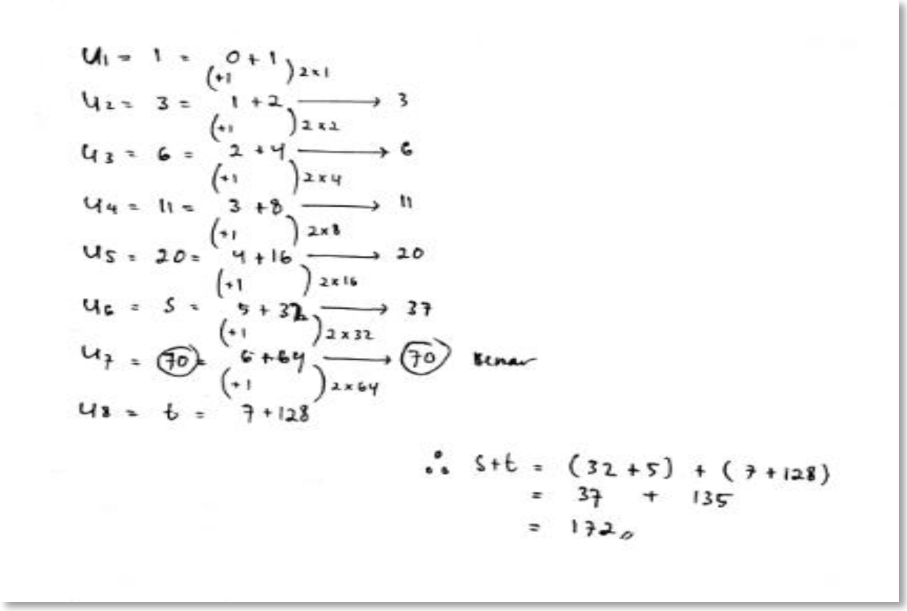


Figure 8.
S3 Written Test Result

Based on the results of interviews and the results of the work done by S3 it is known that in the beginning S3 used an algebraic strategy by using a common variable example that states the terms in order n with $n = 1, 2, 3,$ and so on. The following is an interview transcript showing the S3 thought process in solving research problems using algebraic strategies.

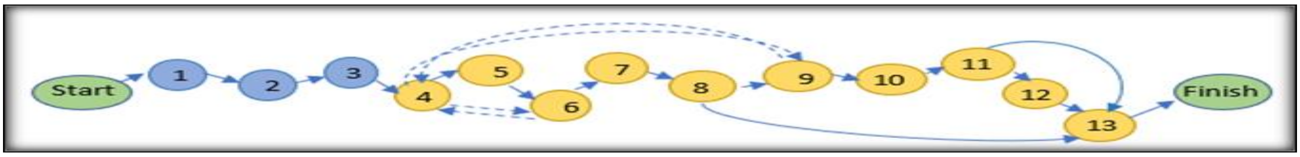
Q 03: Then what was your first thought when you first read about this?

S2 03: This problem is about rows of numbers, ma'am . Then I was told to look for results from $s + t$ because of the sequence of numbers ..., it means you have to find the pattern, Ma'am.

Q 04: What do you do then?

S2 04: This problem asks to look for $s + t \ldots s$ and t in the rows of numbers $1, 3, 6, 11, 20, s, 70, t, \ldots$ This means U_1 equals 1, U_2 equals 3, U_3 equals 6, U_4 equals 11, U_5 equals 20, U_6 equals s, U_7 equals 70, and U_8 equals t. Then I should find the relationship between numbers and the sequence index ... $n = 1$ connected with 1, $n = 2$ connected with 3, $n = 3$ connected with 6, $n = 4$ connected with 11 and so on. But this is difficult ma'am ... so I first changed the numbers in the row of numbers $1, 3, 6, 11, 20, s, 70, t, \ldots$

When having difficulty connecting sequences of numbers and representative numbers, S3 then replaced the strategy used by using arithmetic one. To determine the pattern of a given number, S3 only paid attention to the relationship between numbers without gave regard to its relationship to the sequence of numbers. The generalization form used by S3 to solve the given problem is a symbolic non-formal form. The generalization form used is a number which is the sum of two numbers. The first number follows the patterns 0, 1, 2, 3, ..., and the second number follows the pattern 1, 2, 4, 8, ... S3 cannot determine the general form of a number pattern that can be used to determine terms in a very large order, for example the 100_{th} number of the sequence in the problem. Thus S3 can be categorized as a student with synthetic model generalization. S3 thinking process in resolving given problems which was based on analysis of answers and interview results can be illustrated in the S3 thinking process flow in Figure 9.



Notes:

1. Observing the pattern of numbers: $1, 3, 6, 11, 20, s, 70, t, \ldots$
2. Writing variables that state the term syllable using symbols, for example U_1, U_2, U_3, etc.
3. Making a relationship between variables U1 and 1, U2 and 3, U3 and 6, and so on until connecting U6 and s, U8 and t
4. Manipulating numbers in rows 1, 3, 6, 11, 20, ... into the sum of two integers 1 = 0 + 1, 3 = 1 + 2, 6 = 2 + 4, 11 = 3 + 8, and 20 = 4 + 16
5. Writing a line/sign that indicates the regularity of a number pattern that has been manipulated by giving a hyphen to the first and second set of numbers
6. Paying attention and looking for patterns in the first set of numbers, 0, 1, 2, 3, 4, ...
7. Finding the pattern of relationships between the first number which is having one point bigger than the previous number
8. Supposing that s is the sum of two numbers the first number of which is 5. 70 is the sum of two numbers the first number of which is 6, and t is the sum of two numbers the first number of which is 7
9. Paying attention and looking for patterns in the the second set of numbers, 1, 2, 4, 8, 16.
10. Finding the pattern of relationships between the second number which is having two point bigger than the previous number
11. Supposing that s is the sum of two numbers and the second number of which is $2x16 = 32$. 70 is the sum of two numbers and the second number of which is $2x32 = 64$, and t is the sum of two numbers and the second number of which is $2x64 = 128$.
12. Testing whether the pattern is correct by checking whether 70 is $6 + 64$
13. Expressing her/his belief that $s = 5 + 32 = 37$ and $t = 7 + 128 = 135$ based on the rules found and write $s + t = 37 + 135 = 172$.

Sequenceof Steps
Coordination
Arithmatics Strategy
Algebraic Strategy

Figure 9.
S3 Thinking Process Flow in Solving Problems

In solving problems, S3 used Algebraic strategy from the beginning to the end. The flow of strategy choosing can be seen from Figure 10.

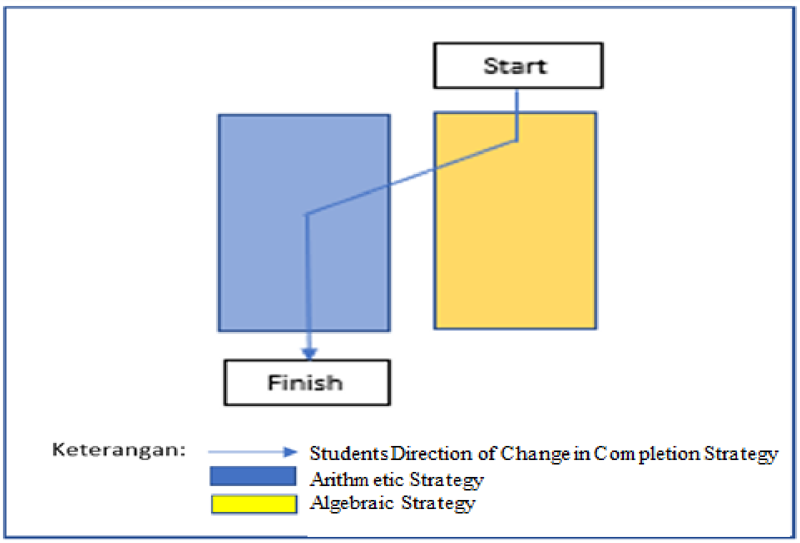


Figure 10.
The Scheme of Changes in the Selection of Generalization Strategies by S3 in Solving Research Problems

Formal Indirect Mental Model.

The written test result of S4 can be seen in Figure 11.

Figure 11.

S4 Written Test Result

Based on the results of interviews and the results of the work done by S4 it is known that in the beginning S4 only paid attention to the arithmetic relationships between numbers in the sequence. The following is an interview transcript that shows S4 thinking process in solving problems using arithmetic strategy.

Q 05 How do you search for rows of patterns 1, 3, 6, 11, 20, s, 70, t, ...?

S4 05 Look, Ma'am, I'll look for the difference between numbers, then I'll see the difference between the two adjacent numbers...

QP 06 Then, what?

S4 06 The difference is this Ma'am, (showing the results of her work), see 2, 3, 5, 9, s-20, 70-s, and t-70. But the difference isn't the same Ma'am ... so I'm confused ...

Furthermore, having been realized that the difference between the two large numbers is different from one another, S4 observed the new sequence of numbers $2, 3, 5, 9, s-20, 70-s, t-70, \ldots$ After making observations, S4 manipulated the numbers in the sequence to find out the pattern. Before carrying out the manipulation process, S4 used an algebraic strategy to simplify the process of completion by assuming $s-20$ with a, $70-s$, with b and $t-70$ with c. Thus S4 had replaced the strategy used by using algebraic strategy. The following is an interview transcript that shows S4 thinking process in searching for a, b and c.

S4 10: Look, ma'am, I noticed the pattern of the difference between $2, 3, 5, 9, a, b, c, \ldots$ this, Ma'am ... I found out that 3 is $2x2-1$, 5 is $2x3-1$, 9 is $2x5 -1$ so if the general formula is made right $U_1 = 2$, $U_2 = 2xU_1 - 1$, $U_3 = 2xU_2 - 1$ and so on or $Un = 2xU_{(n-1)} - 1$ with $U_1 = 2$.

Q 11: Then what?

S4 11: Yes, Ma'am ... after that a, b and c are searched ...

Q 12: How do you find these a, b and c?

S4 12: Um ... because a is U_5 from rows 2, 3, 5, 9, ..., then, a can be known by replacing n in the previous U_n formula (pointing to the general formula $Un = 2xU_{(n-1)} - 1$ with $n = 5$, so later you can get $a = 2x9 - 1 = 17$, if b is replaced n by 6 so $b = 2x17 - 1 = 33$, c can find by replaced n with 7 so $c = 2x33 - 1 = 65$.

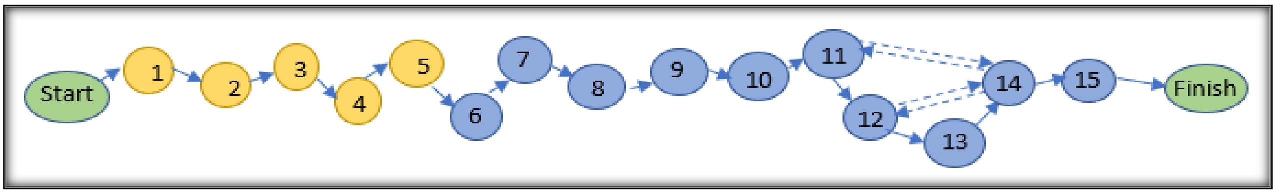
Q 13: Then, what do you do?

S4 13: Yes, then I looked for s and t requested Ma'am... $s = 20 + a = 20 + 17 = 37$, $t = 70 + c = 70 + 65 = 135$ mom ... then $s + t$ means 172 ma'am.

Q 14: Are you sure about this result?

S4 14: Yes, Ma'am, because I think this is correct, Ma'am. I have tried to match if a and b are obtained from the general formula rules $Un = 2xU_{(n-1)} - 1$ with $U_1 = 2$., then $s = a + 20$, and $s + b = 70$ means true and same as what is known in the problem. So $s = 37$ and $t = 135$ must be right ma'am

Based on the interview transcript, it can also be seen that the generalization used by S4 to solve the given problem is a form of symbolic formal generalization eventhough the general formula produced is a recursive formula. Thus S4 is included in students with formal indirect mental model. S4 thinking process in solving problems given which was based on analysis of answers and interview results can be illustrated in S4 thinking process flow in Figure 12.



Notes:

1. Observing the pattern of numbers: $1, 3, 6, 11, 20, s, 70, t, \ldots$
2. Writing curved lines that indicate the pattern of differences between two numbers in rows of numbers $1, 3, 6, 11, 20, s, 70, t, \ldots$
3. Calculating the difference between two numbers that are close together in rows of numbers $1, 3, 6, 11, 20, s, 70, t, \ldots$
4. Stating the difference in rows of numbers $1, 3, 6, 11, 20, s, 70, t, \ldots$, not the same, getting bigger, irregular, and forming a new sequence of numbers, namely: $2, 3, 5, 9, s - 20, 70 - s, t - 70, \ldots$
5. Observing the pattern of numbers $2, 3, 5, 9, s - 20, 70 - s, t - 70, \ldots$
6. Supposing $s - 20$ with variable a; $70 - s$ with variable b; $t - 70$ with variable c
7. Observing the pattern of the new sequence of numbers, $2, 3, 5, 9, a, b, c, \ldots$
8. Manipulating the numbers in rows $2, 3, 5, 9, a, b, c, \ldots$ into multiplication or addition of two integers
9. Writing a line / sign that indicates the regularity of number patterns on rows of numbers $2, 3, 5, 9, a, b, c, \ldots$. that have been manipulated
10. Stating the rows of numbers $2, 3, 5, 9, \ldots$ regularly follow the rules $Un = 2xU_{(n-1)} - 1$ with $U_1 = 2$
11. Writing down the next number of rows of numbers $2,3, 5, 9, \ldots, is\ 17, 33, 65, \ldots$ so that $a = 17$, $b = 33$ and $c = 65$
12. Calculating $s = 20 + a$ and calculating $t = 70 + c$;
13. Writing down and declare $s = 37$, and $t = 135$
14. Expressing confidence in the answer by showing that $70 = s + b$ is true if $s = 37$ and $b = 33$
15. Calculating and writing $s + t = 37 + 135 = 172$

Steps of completion	Coordination
Arithmatics Strategy	Algebraic Strategy

Figure 12.
S4 Thinking Process Flow in Solving Problems

S4 in solving problems uses algebraic strategy from the beginning to the end of completion. The flow of strategy selection carried out by S4 can be illustrated in Figure 13.

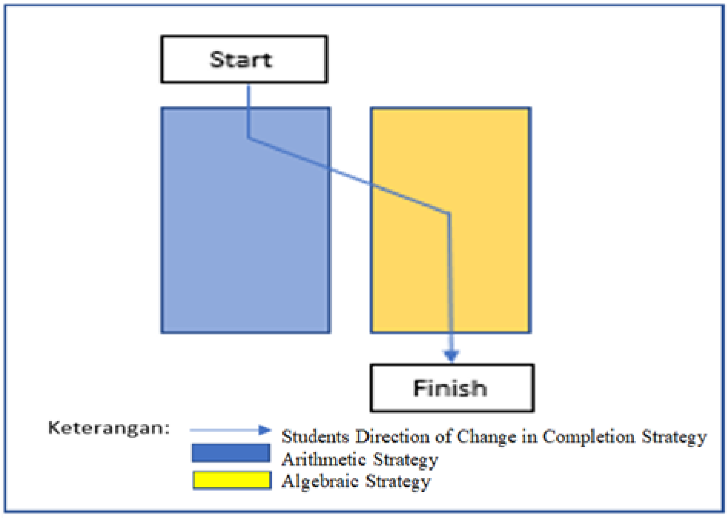


Figure 13.
The Scheme of Changes in the Selection of Generalization Strategies by S4 in Solving Research Problems

Discussion

Students' mental models appear through the selection of generalization strategies used in solving problems. The difference in strategy using can be known when students (1) find number patterns, (2) find relationship between patterns, and (3) make symbolization.

In the process of problems solving all subject contained the activities of observing, questioning, trying, reasoning, and concluding in accordance with the principles of the scientific approach. In the observing, trying, reasoning and concluding stage, all of subject has different technic.

In the process of thinking, students could find certain patterns in accordance with the general stage of APOS, namely the action stage, the stage of the process, the object stage and the stage of the scheme. In solving the given problem that is looking for the number of s and t, the subject of the Direct Mental Model category S1 and S2 in the "action" stage had realized that the numbers arranged in rows form a pattern, so to solve the problem s + t, S1 and S2 should find the applicable rules in the sequence. To generalize the conjecture, S1 and S2 observed the case by repeatedly reading the given problems and observing the numbers in the sequence. Based on observations of the numbers in existing cases, S1 and S2 organized the case by rewriting the problems in the form of a row or making a list to link the sequence of numbers and numbers that are known, making it easier for the subject to work in the case. This is consistent with Allen's opinion (2001) that the most commonly used way of organizing certain case is by listing data or tables.

In the action stage, S1 and S2 observed and read questions carefully and thoroughly, and organized cases that aim to make it easier for subjects to work on cases. This is in accordance with the opinion of Dubinsky & McDonald (2001) that action is a transformation of a thing or object carried out by an individual as an external need both explicitly and from memory, step by step as a guide for conducting operations. Furthermore, Dubinsky & McDonald (2001) stated that interiorization is a change from a procedural activity to be able to do that activity again in the imagination of several notions that affect the resulting conditions. In carrying out interiorization of actions, not only coordinated some actions that have been carried out, but also coordinated with the initial knowledge possessed by students. The initial knowledge possessed by students is like looking for differences between numbers and expressing a number in the sum of two different numbers.

In the object stage, S1 formulated the conjecture by encapsulating the process of seeing the difference or addition between numbers; whereas S2 formulated the conjecture based on the relationship of the sequence index of numbers and patterns resulting from the manipulation of numbers. The method used by S1 and S2 in the process stage used different symbols. The use of different signs and symbols indicates that the symbols formed by the subjects are very meaningful to students and describe their knowledge, according to what Steinbring as stated in Botzer & Yerushalmy (2008) that mathematical signs or symbols are tools for coding and describing knowledge, communicating knowledge and communicating their mathematical knowledge.

In the scheme stage, S1 was sure that the general formula for determining the next number of rows 1, 2, 4, ... is that the next number in rows 1, 2, 4, ... is twice the previous number. Furthermore, to be able to solve the problem given, S1 did calculations to determine s and t based on the rules of number patterns 1, 2, 4, ..., which were believed to be correct. Whereas S2 was sure that rows of numbers 1, 3, 6, 11, 20, ..., follow the rules $U_n = (n-1) + 2^{(n-1)}$, with n expressing the order in the sequence of numbers. S2 determined s and t using rules that have validated validity, and s is the sixth and t is the eighth number of sequence 1, 3, 6, 11, 20, ...

Generalizations made by S1 and S2 are different. S1 produced generalization in the form of arithmetic while S2 produced generalization in the form of algebra symbolically. This is in accordance with the opinion of Radford (2003) that the type of generalization can be either arithmetic or symbolic algebra generalization.

In solving the given problem that is looking for the number of s and t, the subjects of the Indirect Mental Model category, S3 and S4, in the "action" stage have realized that the numbers arranged in rows of numbers form a pattern, so as to solve the problem, S3 and S4 should find the rules that apply in the sequence. In solving the problem of a given number pattern, S3 and S4 used a combination of two generalization strategies described by Radford (2006) and during the problem solving process. To solve this problem, they determined s and t, then looked for the sum of s and t. In the action stage S3 and S4 realized that the sequence of numbers has a certain pattern. To obtain the values expressed by s and t, S3 used the algebra generalization strategy and moved using the arithmetic strategy. Whereas S4 used arithmetic generalization strategy, then moved using algebraic strategy.

In solving the problem of a given number pattern, at the beginning of the completion, S3 used algebraic strategy. It can be seen when S3 organized the case, she had provided algebraic symbolization which states the sequence of numbers in a formal form, while S4 at the beginning of completion used arithmetic strategy by not doing symbolic actions to the numbers in rows. At the end of process S4 used algebraic strategy by starting to use algebraic operations to completion the problems, while S3 used arithmetic strategy by using ordinary arithmetic operations to solve problems. At the stage of formulating and constructing conjecture S4 used algebraic strategy. It generates general rules or formulas that apply to determine the $n-th$ numbers in the form of formal algebraic symbols, whereas S3 used arithmetic strategy to produce a form of rules or formulas to determine the $n-th$ numbers not in formal symbolic form.

There are various causes which might influence the phenomena. The teacher's perception and practice may become the strong cause. Several research suggest that teaching practice have strong link with the students' performance (Wenglinsky, 2002). Teachers who have never taught or made their students engaged with problems may cause their students are not accustomed to deal with problem solving. Furthermore, stimulus from teacher is also important to help the students learn (Hidayah, Pujiastuti, Chrisna, 2017). In a learning by using problem solving approach, teachers need to support and guide their students through scaffolding instruction enable management of learning task (Hillman, 2003).

What we found in this research indicate that teacher has introduced various problems to the students. This is the importance role of teacher to also guide or assist the students to use the heuristic strategies to deal with mathematics problems (Anwar & Rofiki, 2018; Kusdinar et al. 2017). In working with junior high school students, we often find that they have a considerable amount of knowledge about algebra and numbers, and they also know a lot of the strategies for working through problems (Nite, 2017). The key of success in this problem comes when students consider the cognitive evaluations generated during problem solving to be personally important and then when they constructively utilize the differentiated emotions that occur as a result (Carifio, 2015).

The methodological consideration of active learning intervention studies proposes that in the future, researchers should pay attention to the descriptions of study settings (e.g., the number of participants) as well as to how they approach learning outcomes (Hartikainen et al. 2019). The development of higher education requires a systematic approach (Lindblom, 2019).

Conclusion

In summary, the present study found four types of mental models. First, Direct Synthesis Mental Model in solving patterns generalization problems occurs when students resolve problems about number patterns using only arithmetic strategy (Radford, 2006) from the beginning to the end. The final form of generalization used by students in solving problems cannot be stated in formal symbolic form (Vosniadou & Brewer, 1992) which can be applied in larger cases. Second, Direct Formal Mental Model in solving patterns generalization problems occurs when students in solving problems about number patterns only use algebraic strategy (Radford, 2006) from the beginning to the end. The final form of generalization used by students in solving problems can be expressed in a formal symbolic form (Vosniadou & Brewer, 1992) and can be applied in larger general cases. Third, Indirect Synthesis Indirect

Mental Model occurs when students in the process of solving problems about number patterns use a combination of arithmetic and algebraic generalization strategies alternately (Radford, 2006; Prayekti et al. 2019). During the process of solving problems, students make changes in problem solving strategies from algebraic generalization strategy to arithmetic generalization strategy so that the resulting rules only pay attention to arithmetic relationships between numbers. The final form of generalization used by students in solving problems cannot be stated in formal symbolic form which can be applied in larger cases (Vosniadou & Brewer, 1992). Fourth, Indirect Formal Mental Switch Model occurs when students in the process of solving problems about number patterns use a combination of arithmetic generalization strategy and algebraic generalizations strategy alternately (Radford, 2006; Prayekti et al. 2019). During the process of solving a problem, students make a problem solving strategy from an arithmetic generalization strategy to an algebra generalization strategy so that the resulting rules pay attention to the relationship between numbers and substitute variables that indicate the sequence of numbers in the sequence. The final form of generalization used by students in solving problems can be expressed in a formal symbolic form (Vosniadou & Brewer, 1992) and can be applied in general and larger cases.

Limitations of Research and Recommendations

The students' critical thinking process in solving problems was reviewed based on APOS theory. Through APOS theory, students could construct a concept then it was expressed into spoken and written communications. Font et al. (2016) stated that implementation of APOS theory in learning mathematics could construct students' understanding or mental model.

In this research, the subjects were limited on junior high school students. Further research is suggested to have senior high school students or higher education students and in another mathematics topic. It is suggested for further researches to investigate students' mental models failure in solving mathematics problems.

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